On Hausdorff dimension.

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Def α-dimensional Hausdorff content is defined as

$$H_\alpha(K) := \inf\{\sum \varepsilon_j^\alpha : K \subset \cup K_j,\ \operatorname{diam} K_j = \varepsilon_j\}$$

Can even generalize it slightly. Let $h(t) \geq 0$ be a strictly increasing continuous function on $\mathbb{R}_+$, $h(0)=0$. Gauge function. Define h Hausdorff content as

$$H_h(K) := \inf\{\sum h(\varepsilon_j) : K \subset \cup K_j,\ \operatorname{diam} K_j = \varepsilon_j\}$$

Same as what we had before for $h(t)=t^\alpha$.

Note that if K is countable, $H_h(K)=0$ (because for any ε we can cover $a_i \in K$ by disk of diameter ε_i with $h(\varepsilon_i) < 2^{-i}\varepsilon$.

Lemma 1. If $H_h(K)=0$ and $\varlimsup_{t\to 0} \frac{g(t)}{h(t)} < \infty$, then $H_g(K)=0$.

Proof $\forall \varepsilon>0\ \exists$ covering K_i of K such that $\sum h(\operatorname{diam} K_i) < \varepsilon$.

Then, since h is strictly increasing, $\max \operatorname{diam} K_i \to 0$ as $\varepsilon \to 0$. Thus for some C, $g(\operatorname{diam} K_i) < C h(\operatorname{diam} K_i)$, so $\sum g(\operatorname{diam} K_i) < C\varepsilon \Rightarrow H_g(K) < C\varepsilon$ ■

Corollary 2. If $H_\alpha(K)=0$ and $\beta>\alpha$ then $H_\beta(K)=0$.
If $H_\alpha(K)>0$ and $\beta<\alpha$, then $H_\beta(K)>0$.

Similarly to the discussion of (lower) Minkowski dimension, can now define Hausdorff dimension as

$$\operatorname{Hdim} K := \inf\{\alpha : H_\alpha(K)=0\} = \sup\{\alpha : H_\alpha(K)>0\}$$

One problem with H_h - it is not a measure.

Example. $H_{1/2}([0,1)) = H_{1/2}([1,2)) = 1$, but $H_{1/2}([0,2)) = \sqrt{2} < H_{1/2}([0,1)) + H_{1/2}([1,2))$. (simply because $a^{1/2}+b^{1/2} \geq (a+b)^{1/2}$)

To make it into a measure, force covering by smaller and smaller sets:

$$m_h^\varepsilon(K) := \inf\{\sum h(\varepsilon_j) : K \subset \cup K_j,\ \operatorname{diam} K_j = \varepsilon_j < \varepsilon\}$$

and

$$m_h(K) := \lim_{\varepsilon\to 0} m_h^\varepsilon(K)$$

The limit always exists (as a limit of an increasing function), but can be infinite.

Thm m_h is a measure (usually, not even σ-finite!)

Property $m_h(K) \geq H_h(K)$ and $H_h(K)=0 \Leftrightarrow m_h(K)=0$

Proof The first statement follows from the definition.

For the second notice that if K_i is a covering such that $\sum h(\operatorname{diam} K_i) < \varepsilon$ then $\max \operatorname{diam} K_i < h^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$. So $m_{h,h^{-1}(\varepsilon)}(K) < \varepsilon$. Let $\varepsilon \to 0$. ■

Lemma. If h, g are two gauge functions, $\lim_{t\to 0}\frac{h(t)}{g(t)}=0$, and $m_g(K)<\infty$ then $m_h(K)=0$

Pf. Fix any $\delta>0$ and choose $\varepsilon>0$ such that $t<\varepsilon \Rightarrow h(t) < \delta g(t)$.
Consider a covering of K by K_i such that $\operatorname{diam} K_i < \varepsilon$ and $\sum g(K_i) < m_{g,\varepsilon}(K) + \varepsilon$. Then

$$m_{h,\varepsilon}(K) \leq \sum h(K_i) < \delta \sum g(K_i) < \delta(m_{g,\varepsilon}(K)+\varepsilon).$$

Now let $\varepsilon \to 0$ to see that $m_h(K) < \delta$. ■

Corollaries:

1) If $\lim_{t\to 0}\frac{h(t)}{g(t)}=0$, $m_h(K)>0$, then $m_g(K)=\infty$

2) If $\alpha = \operatorname{Hdim} K$ and $\beta<\alpha<\gamma$, then $m_\beta(K)=\infty$ and $m_\gamma(K)=0$

Thus $\operatorname{Hdim}(K) = \inf\{\alpha : m_\alpha(K)=0\} = \sup\{\alpha : m_\alpha(K)=\infty\}$.

So we can estimate the Hausdorff dimension from above, by presenting a cover. How to estimate it below?

Def. A measure μ is called *h-smooth* if for some C and for every ball $B(x,r)$, $\mu(B(x,r)) \leq Ch(r)$.

Thm (Mass distribution principle). Let $\mu(K) > 0$ for some h-smooth measure, then $m_h(K) \geq H_h(K) \geq \frac{\mu(K)}{C}$, where C is the constant in the definition of h-smoothness.

Proof. Let (K_i) be any cover of K. Then $K_i \subset B(x_i, \operatorname{diam} K_i)$ for any $x_i \in K_i$. Then $\mu(K) \leq \sum \mu(K_i) \leq \sum C h(\operatorname{diam} K_i)$. Take inf over all the coverings. ∎

Corollary. If $\mu(K) > 0$ for some α-smooth measure ($\mu(B(x,r)) \leq Cr^{\alpha}$) then $\operatorname{Hdim} K \geq \alpha$.

Using this, it is easy to prove that $\operatorname{Hdim} C = \frac{\log 2}{\log 3}$ (C - the usual Cantor set).
Construct μ by assigning $\mu(I_n^k) = 2^{-n}$ for any interval $I_n^k \in C_n$. $\mu(C) = 1$, and notice that for $3^{-n} \leq r < 3^{-n+1}$, $B(x,r)$ intersects at most one I_{n-1}^k, so $\mu(B(x,r)) \leq 2^{-n+1} \leq 2r^{\frac{\log 2}{\log 3}}$, so μ is $\frac{\log 2}{\log 3}$-smooth.
Thus $\frac{\log 2}{\log 3} \leq \operatorname{Hdim} C \leq \operatorname{Mdim} C = \frac{\log 2}{\log 3}$.

Def Dimension of a measure: μ - Borel measure in $\mathbb{R}^d$.
$\dim \mu := \inf \{\operatorname{Hdim} A : \mu(A^c) = 0,\ A \subset \mathbb{R}^n \text{ - Borel}\}$.
Another, equivalent, def
$\dim \mu := \inf \{\alpha : \mu \perp m_\alpha\}$

Lower dimension of a measure:
$\underline{\dim}\, \mu := \inf \{\operatorname{Hdim} A : \mu(A) > 0,\ A \subset \mathbb{R}^n \text{ - Borel}\}$.